

THE NATURE OF VERTICAL NATURAL CONVECTION FLOWS RESULTING FROM THE COMBINED BUOYANCY EFFECTS OF THERMAL AND MASS DIFFUSION

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Abstract—This paper concerns the laminar flows which arise in fluids due to the interaction of the force of gravity and density differences caused by the simultaneous diffusion of thermal energy and of chemical species. Species concentration levels are assumed small, as is typical for many processes in water and in atmospheric air. The usual Boussinesq approximations yield a set of equations which are shown to have solutions of similarity form for combined buoyancy effects, for vertical flows adjacent to surfaces and in plumes. This similarity is of the same form as that found for single buoyancy mechanism flows. The resulting equations were integrated for air and water for various practical values of the Schmidt number and for multiple buoyancy effects aiding and opposing. The results show many interesting effects on velocity, heat and mass transfer, and on laminar stability. A comparison of the results with those of integral method analysis shows the limits and reasons for failure of these approximate calculations in the more complicated of such combined buoyancy mechanism flows.

NOMENCLATURE

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|-------------|--|-------------|---|
| C , | nondimensional species concentration; | t , | fluid temperature; |
| C_p , | specific heat of the fluid at constant pressure; | u , | velocity component in x direction; |
| D , | chemical molecular diffusivity; | v , | velocity component in y direction; |
| f , | nondimensional stream function; | x , | vertical distance along the surface; |
| g , | gravitational acceleration; | y , | horizontal distance from the surface; |
| Gr , | Grashof number defined by equations (12); | α , | thermal molecular diffusivity; |
| k , | thermal conductivity of the fluid; | β , | volumetric coefficient of thermal expansion; |
| Le , | Lewis number = α/D ; | β^* , | volumetric coefficient of expansion with concentration; |
| M , | mass convection rate of chemical species; | ϕ , | nondimensional temperature; |
| \dot{m} , | vertical mass flow per unit length; | μ , | dynamic viscosity of the fluid; |
| \dot{M} , | vertical momentum flow per unit length; | ν , | kinematic viscosity of the fluid; |
| N , | parameter defined by equation (13); | ψ , | stream function; |
| N_C , | parameter defined by equation (24); | η , | similarity variable; |
| N_r , | parameter defined by equation (25); | ρ , | fluid density. |
| n , | exponent defined by equation (10); | | |
| P , | constant: scaling factor; | | |
| Pr , | Prandtl number = ν/α ; | | |
| Q , | constant: scaling factor; | | |
| \dot{Q} , | convection rate of thermal energy; | | |
| Sc , | Schmidt number = ν/D ; | | |

Subscripts

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|------------|--|
| C , | based on species concentration level; |
| x , | based on distance x ; |
| t , | based on temperature level; |
| 0 , | at the surface or at the plume midplane; |
| ∞ , | in the undisturbed fluid. |

INTRODUCTION

THERE are many transport processes which occur in nature and in the devices of man in which flow is driven or modified by density differences caused by temperature, chemical composition differences and gradients, and material or phase constitution. Such processes are very important and probably are the most important ones which we must understand.

Processes in which the buoyancy driving force arises solely due to temperature difference have received considerable attention for both steady and transient internal and external flows, both laminar and turbulent, for stability and transition, and with many additional imposed conditions and effects. Many of the most important general characteristics and mechanisms of such flows have been clarified.

However, flows arising from differences in concentration or material constitution alone and in conjunction with temperature effects, have received little attention. This is unfortunate since such buoyancy effects occur in many processes. Clearly atmospheric flows, at all scales, are driven appreciably by both temperature and H_2O concentration differences. Flows in bodies of water are driven through the comparable effects upon density of temperature, concentration of dissolved materials, and suspended particulate matter.

The purpose of the present study is to investigate flows resulting from buoyancy forces which arise from a combination of temperature and species concentration effects of comparable magnitude. This circumstance arises often, especially in the natural environment. There are many interesting aspects of such flows, for example, resulting transport characteristics, results of the opposition of the two effects, influence of the combined effect on the stability of laminar flows, and the effects of values of the relative transport parameters the Prandtl and Schmidt numbers. The results presented below consider all of these aspects for laminar boundary region flow in freely rising convection plumes and for flows adjacent to vertical surfaces.

Processes involving "mass" transfer effects have long been recognized as important, principally in chemical processing equipment. There is an extensive literature concerning overall behavior and design considerations. However, there is not much information available concerning the detailed nature of flow and transport for well defined natural convection flow circumstances.

One of the earliest such studies known to us is that of Somers [1], which is concerned with the combined thermal and species diffusion driven flow that would arise adjacent to a wetted isothermal vertical surface in a non-saturated atmosphere. The condition of very small diffusing species concentration was used and an integral method analysis was carried out for uniform surface temperature and diffusing species concentration. The principal results were a transport relation and the indication that a combined driving force might be written in which the species diffusion contribution is modified by the square root of the Lewis number, i.e. \sqrt{Le} . This latter result comes from arguments concerning relative nominal boundary region thickness. The analysis is expected to be reasonable only around Prandtl and Schmidt numbers of 1.0, with one buoyancy effect being very small compared to the other.

Mathers *et al.* [2] formulated the same problem in terms of the boundary layer differential equations resulting from force-momentum, energy, and chemical species conservation, at very low concentration. Neglecting inertia effects (this is not generally permissible even at very large Prandtl numbers in purely thermally driven natural convection flow) the resulting equations were solved on an analogue computer for $Pr = 1.0$ and $Sc = 0.5-10$ for the ratio of species and thermal diffusion buoyancy effects of 1.0 and 0.5. Resulting transport information appears to support the \sqrt{Le} factor of Somers. Experimental results with naphthalene and benzene diffusing from a sphere in air were grouped to within perhaps 20% spread in terms of the variables used.

An integral analysis by Wilcox [3] for uniform conditions along a surface, using arbitrarily assumed velocity, temperature and concentration distributions, led to transport equations which followed Somers in assigning a factor \sqrt{Le} to the species diffusion induced component of buoyancy. The general transport predictions are compared with exact results for flows driven entirely by thermal diffusion. Differences are appreciable and the comparison does not validate the use of a simple integral method analysis for convection with combined buoyancy effects. A particular result is calculated for an adiabatic surface.

Gill *et al.* [4] analysed the combined buoyancy effects in gaseous binary diffusion and include the effects of varying transport properties and species enthalpies while neglecting the phenomena of species thermal diffusion and diffusion thermal energy. This study concerns combined thermal and species diffusion in gases when the concentrations of the species are comparable. The appropriate surface condition of appreciable normal velocity was used. Numerical results were obtained which indicate the relative importance of the various effects.

Lowell and Adams [5] consider the question of possible similarity solutions for combined buoyancy mechanism flows formulated with the limitations assumed by Gill *et al.* [4]. Constant properties were assumed. The boundary condition of appreciable normal velocity at the surface and the effects of comparable concentration levels were included. The only similarity condition found was that of an isothermal surface. Adams and Lowell [6] present results of a numerical analysis of the above similarity formulation for a subliming organic surface in air. They note numerical difficulties when the buoyancy effects are opposed and call them flow instabilities, without any apparent justification.

Cardner and Hellums [7] formulated the combined buoyancy mechanism flow, using the low species concentration level approximation. They analyzed flow adjacent to a liquid layer, assumed to be isothermal. Interfacial velocity

was taken into account and some numerical results were presented.

Lightfoot [8] considered asymptotic behavior for $Sc \gg Pr$ and for the species diffusion buoyancy effect very small compared to that caused by thermal diffusion. The resulting transport relation was not in agreement with the general result of Wilcox, specialized to this circumstance.

Adams and McFadden [9] report experimental measurements of the heat and mass transfer parameters during the sublimation of *p*-dichlorobenzene ($Sc = 2.23$) from a heated vertical surface into air. The two buoyancy effects were opposed. Presentation of the data in terms of the effective net buoyant force predicted by Somers resulted in experimental data being 10–15 per cent lower than any of the integral method results. However, the data did correlate well when the factor \sqrt{Le} was omitted in calculating the net buoyant force.

Den Bouter *et al.* [10] report an experimental study of simultaneous thermal and chemical species diffusion by an electrochemical method, between a vertical copper plate maintained at constant temperature and a copper sulphate-sulphuric acid solution. Measurements were made with the two buoyancy effects aiding and opposed. The mass and heat transfer parameters, correlated in terms of a combined buoyancy effect, calculated with the \sqrt{Le} term, agree well with a single curve for the effects aiding. However, for the effects opposed, the disagreement from the single curve is random and up to over 30 per cent in magnitude. Perhaps even that level of agreement is surprising, since values $Pr \approx 10$ and $Sc \approx 2000$ apply in this circumstance.

Manganaro and Hanna [11] discuss the equations which would govern simultaneous thermal and chemical species transport in a binary gas mixture (for important binary diffusion effects). Normal components of surface velocity are accounted for as are variable properties. Both forced and natural convection flows are considered.

Saville and Churchill [12] consider simultaneous thermal and species diffusion caused buoyancy forces for negligible interfacial velocity and for very low concentration level diffusion, using the formulation of Saville and Churchill [13] for natural convection adjacent to general surfaces. Attention is restricted to asymptotic processes in terms of the Prandtl and Schmidt numbers. The dependence of the transport parameters are found for $Pr = Sc \rightarrow 0$, for $Pr = Sc \rightarrow \infty$, for $Pr \rightarrow 0$ and $Sc \rightarrow \infty$, and for Pr and $Sc \rightarrow \infty$ with $Sc \gg Pr$. For purely thermal diffusion caused buoyancy, the Prandtl number dependences are the well-known ones and the species diffusion effect is asymptotically dependent upon the Schmidt number in the same way. These results are principally interesting here in showing the relative species diffusion effect in the asymptotic processes for $Pr = Sc$.

The foregoing discussion and summary of principal past studies of flows and transport arising from multiple buoyancy causing agencies indicate that available information stems mainly from integral method analysis or concerns asymptotic processes. The integral method results follow from somewhat arbitrary assumptions and formulations and are at variance with experimental and more exact analytical results. They are not known to provide a reliable guide for processes with widely different Prandtl and Schmidt number nor for a large difference in aiding and opposed effects. The asymptotic results are principally useful for $Pr = Sc$.

Considering the importance of these combined buoyancy effect flows, it seems desirable to consider in detail formulation in terms of the relevant equations and to calculate the flow and transport characteristics over a range of the relevant parameters. After casting such flows in the context of a general formulation it is a direct extension to treat other more complicated questions such as plume flows, laminar stability, flow transients, transition, inhomogeneous boundary conditions, etc. The present work considers flows with combined buoyancy effects for a range of relative values thereof, for several

Prandtl numbers for the relevant practical ranges of Schmidt number. The nature of the laminar stability limits is discussed. The analysis is restricted to chemical species diffusion processes for which the concentration level is very low, for which interfacial velocities may be neglected, and for which species thermal diffusion and diffusion thermal energy effects are negligible. This set of assumptions is reasonable in most terrestrial processes in air and in water, the concentrations and differences being small, though often important in causing buoyancy forces.

ANALYSIS

The relevant equations result from conservation conditions of mass, of force-momentum, of energy, and of the diffusing molecular species. These equations are simplified for typical natural convection flows by the Boussinesq approximations, which apply for small diffusion caused density variations. This simplification applies to the continuity equation (which becomes the constant density form) and to the evaluation of the buoyancy force arising from density differences. The set of equations are written below in terms of fluid velocity \tilde{V} , temperature t , concentration of a single diffusing species C , gravitational force $\rho\tilde{g}$, and pressure p . The molecular transport properties are μ , k and D . The concentration equation is written in the form appropriate when the concentration C of the diffusing species is very small compared to that of the other chemical species present, for example, water vapor, carbon dioxide, or oxygen diffusing in atmospheric air or salinity diffusing in water. The quantity Φ is the conventional function associated with the thermal effect of the viscous dissipation of energy.

$$\nabla \cdot \tilde{V} = 0 \quad (1)$$

$$\rho \frac{D\tilde{V}}{D\tau} = \rho\tilde{g} - \nabla p + \mu\nabla^2\tilde{V} \quad (2)$$

$$\rho C_p \frac{Dt}{D\tau} = k\nabla^2 t + \beta T \frac{Dp}{D\tau} + \mu\Phi \quad (3)$$

$$\frac{DC}{D\tau} = D\nabla^2 C. \quad (4)$$

These equations ignore the secondary phenomena of species thermal diffusion and diffusion thermal energy effects.

Additional approximations and formulations remain to be made. The second part of the Boussinesq approximation concerns the combination $\rho\tilde{g} - \nabla p$ in the force-momentum balance. For external natural convection flows, that is, concentrated convection flows rising in an extensive ambient medium, which is perhaps stably stratified in the direction of \tilde{g} , it is convenient to dissociate the actual pressure into two components, the local hydrostatic value $p_h(\nabla p_h = \rho_\infty\tilde{g})$, and the pressure difference in the convection region associated with motion, p_m .

Consider then the term $\rho\tilde{g} - \nabla p_h = \tilde{g}(\rho - \rho_\infty) = g(\rho_\infty - \rho)$, where g is the magnitude of the gravity field, downward in a coordinate system taken positive upward, with reduced density assumed generated over most of the flow region as the net effect of thermal and mass diffusion. A series expansion of $\rho_\infty - \rho$ in terms of t , p and C , at a given elevation indicates that the p effect may be neglected and that only the linear term of the temperature effect need be retained for $\beta\Delta t \ll 1$, where $\beta = -(1/\rho)(\partial\rho/\partial T)_{p,C}$ is the volumetric coefficient of thermal expansion (see Gebhart [14]). A similar argument indicates that only the linear term of the concentration effect need be retained for $\beta^*\Delta C \ll 1$ where β^* is the volumetric coefficient of expansion with concentration, that is $-(1/\rho)(\partial\rho/\partial C)_{T,p}$. This latter condition on concentration differences is met in most atmospheric and oceanic flows. Thus, the term $g(\rho_\infty - \rho)$ is simplified to the following.

$$g(\rho_\infty - \rho) = g\rho\beta(t - t_\infty) + g\rho\beta^*(C - C_\infty) \quad (5)$$

where, for example, for ideal gas behavior

$$\beta^* = \left[\frac{MW_a}{MW_C} - 1 \right] \frac{1}{\rho}$$

see Saville and Churchill [12]. MW_C means

molecular weight of diffusing species and MW_a refers to the other component.

The second component of the total buoyancy term above is often an important effect in flows of interest. The ratio of the two effects for a temperature level of 20°C with a difference of say 10°C, and with the difference in water vapor concentration which accompanies saturated conditions for these temperature levels, is

$$\frac{\beta^*(C - C_\infty)}{\beta(t - t_\infty)} \approx 3.$$

Similarly, for a salinity difference of 100 ppm in sea water with an associated temperature difference of 1°C the ratio is about 0.2. Thus, it is seen that, even for very ordinary processes at low concentration differences, the buoyancy effect of species diffusion may also be important in generating motion.

Remaining approximations to the above general equations concern the last two terms of the energy equation. The pressure term $\beta T(Dp/D\tau)$ may be important for gases, for which $\beta = 1/T$. However, in general for flows of small vertical extent this term may be neglected. The viscous dissipation term is also negligible for flows of small vertical extent for gravity or acceleration fields of the order of terrestrial gravity (see Gebhart [14]).

The present study of laminar flows and of their stability concerns the simple case of two-dimensional flow $\vec{V} = (u, v)$. In terms of coordinate x , vertical and opposed to the direction of the gravity force, and y , normal thereto, the equations are given below with all approximations discussed above. Surfaces and flows inclined to the direction of the gravity force are thus excluded but may be treated by more detailed methods. The equations are written in the boundary layer form appropriate when the thickness of the flow region (in y) is small compared to the distance x above the initiation of the natural convection flow. The requirement for this condition in the combined momentum, thermal, and mass diffusion circumstance will

be indicated subsequently.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = v \frac{\partial^2 u}{\partial y^2} + g\beta(t - t_\infty) + g\beta^*(C - C_\infty) \quad (6)$$

$$u \frac{\partial t}{\partial x} + v \frac{\partial t}{\partial y} = \alpha \frac{\partial^2 t}{\partial y^2} \quad (7)$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D \frac{\partial^2 C}{\partial y^2} \quad (8)$$

The dependent variables of velocity, temperature and concentration are generalized in terms of the stream function and differences $t_0 - t_\infty$ and $C_0 - C_\infty$ between the surface and distant medium locally at x . They are written in terms of the similarity variable η . The following representation for vertical flows applies for flows arising adjacent to surfaces as well as those rising freely.

$$\psi(x, y) = vc(x) f(\eta), \quad \phi(\eta) = \frac{t - t_\infty}{t_0 - t_\infty},$$

$$C = \frac{C - C_\infty}{C_0 - C_\infty}, \quad \eta = b(x)y. \quad (9)$$

The functions $c(x)$ and $b(x)$ may be found to yield ordinary differential equations for the assigned quantities $(t_0 - t_\infty)$ and $(C_0 - C_\infty)$ of power law Nx^n or of exponential Me^{sx} dependence on x . For each circumstance, certain variations of t_∞ and C_∞ (stratification) also permit similar solutions, as do also certain flows resulting in appreciable viscous dissipation and having appreciable density gradients resulting from hydrostatic pressure variation in gases. The general formulation and discussion of solutions is given in detail by Gebhart [14].

In this study we neglect stratification, viscous dissipation, and other additional effects. Considering among these simpler flows only power law variations,

$$t_0 - t_\infty = N_t x^n, \quad C_0 - C_\infty = N_c x^m \quad (10)$$

it may be shown that the simplest formulation requires for similarity that $n = m$ and that n may have any value. However, physical realism with respect to leading edge and local flow requires $-0.6 \leq n < 1$.

For this dependence the values of the functions for combined thermal and chemical species diffusion may be written as:

$$b(x) = \frac{G}{4x} \text{ and } c(x) = G.$$

$$G = 4 \sqrt[4]{\left(\frac{PGr_{x,t} + QGr_{x,c}}{4} \right)} \quad (11)$$

where P and Q are constants and the two local Grashof numbers are:

$$Gr_{x,t} = \frac{g\beta x^3(t_0 - t_\infty)}{v^2} \text{ and}$$

$$Gr_{x,c} = \frac{g\beta^* x^3(C_0 - C_\infty)}{v^2}. \quad (12)$$

We note here that similarity also results from any linear combination of $Gr_{x,t}$ and $Gr_{x,c}$ in equation (11). In fact, each $Gr_{x,t}$ and $Gr_{x,c}$ may be the combination of many different effects some perhaps aiding others opposing. The constants P and Q are only limited by the condition that the linear combination be greater than zero. A useful relation is:

$$N = \frac{Gr_{x,c}}{Gr_{x,t}} = \frac{\beta^*(C_0 - C_\infty)}{\beta(t_0 - t_\infty)}. \quad (13)$$

This quantity measures the relative importance of chemical and thermal diffusion in causing the density difference which drives the flow. It is to be noted that N is zero for no species diffusion, infinite for no thermal diffusion, positive for both effects combining to drive the flow, and negative for the effects opposed.

The coordinate x has its origin at the leading edge, for the configuration of interest, and increases away from it. Reasonable similar flows are restricted to those for which net flow is zero at the leading edge and the results may be applied to circumstances in which this condition

is met. For $P = Q = 1$ this is ensured if $Gr_{x,t} + Gr_{x,c}$ is positive and the leading edge is at the bottom of the surface of interest. For the Schmidt and Prandtl numbers equal, P must equal Q and the limit is $N > -1$. However, for $Pr \neq Sc$ the condition on N is not always this stringent. The quantity $Gr_{x,t} + Gr_{x,c}$ may be negative and still result in net positive buoyancy effects and flow over all or most of the boundary region. This condition results for $Sc > Pr$ for $Gr_{x,c} < 0$ and $Pr > Sc$ for $Gr_{x,t} < 0$. The difficulty of sign in equation (11) is overcome in the choice of P and Q in $PGr_{x,t} + QGr_{x,c}$. This matter is considered in detail in the discussion of the results for $Pr \neq Sc$.

Note that if the flow is driven entirely by species diffusion, G above involves only $Gr_{x,c}$. We note also, that if there is simultaneous diffusion of multiple chemical species, at sufficiently low concentration level, G contains an additional term for each effect and additional species diffusion equations arise.

The differential equations for f , ϕ and C in terms of these variables are:

$$f''' + (n+3)ff'' - (2n+2)f'^2 + \frac{\phi + NC}{P + QN} = 0 \quad (14)$$

$$\phi'' + Pr[(n+3)f\phi' - 4nf'\phi] = 0 \quad (15)$$

$$C'' + Sc[(n+3)fC' - 4nf'C] = 0. \quad (16)$$

In this work we will consider, among the power law variations, the uniform temperature surface condition ($n = 0$) and also the freely rising plume flows ($n = -\frac{3}{2}$) which result solely from thermal and/or chemical species diffusion. The boundary conditions for the above equations for the surface flow are:

$$f'(0) = f(0) = f'(\infty) = 1 - \phi(0) = 1 - C(0) = \phi(\infty) = C(\infty) = 0 \quad (17)$$

where the normal velocity component v_0 at $\eta = 0$ associated with the chemical diffusion process has been assumed to be negligibly small. The condition for this assumption, for the thermal

buoyancy effect dominant, is

$$\frac{v_0 x}{v} \ll \sqrt[4]{\left[\frac{g\beta x^3(t_0 - t_\infty)}{v^2} \right]} \quad \text{or}$$

$$\frac{1}{Sc} \frac{(C_0 - C_\infty)}{\rho} [-C'(0)] \ll 1.$$

For a purely diffusion driven plume flow the boundary conditions are:

$$f''(0) = f(0) = f'(\infty) = \phi'(0) = C'(0) = 1 - \phi(0) = 1 - C(0) = 0. \quad (18)$$

These latter boundary conditions are in the form shown to be optimum by Gebhart *et al.* [15] under the condition that $n = -\frac{3}{2}$. For this circumstance equations (15) and (16) may be simply integrated once in η to lower the order of the problem.

An interesting aspect of such combined diffusion flows arises if the Prandtl and Schmidt numbers are equal. In this circumstance equations (14) with $P = Q$, (15) and (16), along with appropriate boundary conditions, indicate that $\phi(\eta) = C(\eta)$. Thus, only one of equations (15) and (16) are necessary and the buoyancy force term in equation (14) becomes either ϕ or C . The solutions then (f , ϕ and C) are merely those for a single buoyancy effect. The solutions for f and ϕ for several different values of n are known over a very wide range of Prandtl number. The combined effect is then present in the interpretation of the results in that $b(x)$ and $c(x)$ depend upon both $Gr_{x,t}$ and $Gr_{x,c}$. It may also be shown that for the Prandtl and Schmidt numbers different and for flow adjacent to a vertical surface and $n = 0$ that:

$$-\phi' = K(-C') \frac{Pr}{Sc}$$

where K is to this point an undetermined function of Pr , Sc and N .

For the flow adjacent to a vertical surface the local heat transfer parameter, the Nusselt number $Nu_{x,t}$ and the local chemical species transfer parameter $Nu_{x,c}$ (or the local Sherwood

number, Sh_x) are found to be:

$$Nu_{x,t} = \frac{h_{x,t}x}{k} = -\frac{\phi'(0)}{\sqrt{2}} \sqrt[4]{(PGr_{x,t} + QGr_{x,c})} = -\frac{\phi'(0)}{\sqrt{2}} \sqrt[4]{(Gr_{x,t})} \sqrt[4]{(P + QN)} \quad (19)$$

$$Nu_{x,c} = \frac{h_{x,c}x}{D} = -\frac{C'(0)}{\sqrt{2}} \sqrt[4]{(PGr_{x,t} + QGr_{x,c})} = -\frac{C'(0)}{\sqrt{2}} \sqrt[4]{(|Gr_{x,c}|)} \sqrt[4]{\left| \frac{P + QN}{N} \right|} \quad (20)$$

Note that the absolute values in equation (20) are necessary since $Gr_{x,c}$ (and N) may be negative. Thus, the differences in transport are seen to be due to $\phi'(0)$, $C'(0)$ and N . For $Pr = Sc$ the differences are due entirely to N since $\phi'(0) = C'(0)$. The tangential component of velocity u is in general

$$\frac{ux}{2v} = \frac{2}{\sqrt{2}} \sqrt[4]{(PGr_{x,t} + QGr_{x,c})} f' = \frac{2}{\sqrt{2}} (Gr_{x,t}) \sqrt[4]{(P + QN)} f' \quad (21)$$

This value depends upon Pr , Sc , N and $Gr_{x,t}$.

For plume flows the quantities of interest are somewhat different, being principally the temperature and concentration levels in the plume and their change with elevation. Following the formulation of Gebhart *et al.* [15] the relations for these quantities, evaluated from functions f , ϕ , C and N , in terms of plume convection rate of thermal energy \dot{Q} and species M are:

$$t - t_\infty = N_t x^{-\frac{1}{2}} \phi(\eta) \quad (22)$$

$$C - C_\infty = N_c x^{-\frac{1}{2}} C(\eta) \quad (23)$$

where

$$N_t = \left[\frac{\dot{Q}^4}{4^3 g \beta \rho^2 \mu^2 C_p^4 I_t^4} \right]^{\frac{1}{2}} (P + QN)^{-\frac{1}{2}} \quad (24)$$

$$N_c = \left[\frac{NM^4}{4^3 g \beta^* v^2 I_c^4} \right]^{\frac{1}{2}} (P + QN)^{-\frac{1}{2}} \quad (25)$$

and the quantities I_t and I_c are defined as:

$$I_t = \int_{-\infty}^{\infty} f' \phi d\eta \text{ and } I_c = \int_{-\infty}^{\infty} f' C d\eta.$$

The mass and momentum convection rate \dot{m} and \dot{M} in the plume are:

$$\dot{m} = 4^{\frac{3}{2}} J \left[\frac{g \beta \rho^2 \mu^2 \dot{Q}}{C_p I_t} \right]^{\frac{1}{2}} x^{\frac{1}{2}} (P + QN)^{\frac{1}{2}} \quad (26)$$

and

$$\dot{M} = 8L \left[\frac{g \beta \dot{Q} \rho^3 v^3}{C_p I_c 4^{\frac{3}{2}}} \right]^{\frac{1}{2}} x^{\frac{1}{2}} (P + QN)^{\frac{1}{2}} \quad (27)$$

where

$$J = \int_{-\infty}^{\infty} f' d\eta \text{ and } L = \int_{-\infty}^{\infty} f'^2 d\eta.$$

Natural convection flows driven by a combination of multiple diffusion effects are very important in many applications. The foregoing formulations may be analysed to indicate the

Table 1. Schmidt number and buoyancy parameter for various species at low concentration in air and in water at approximately 25°C and 1 atmosphere

	$Sc = v/D$	$\rho\beta^* = -(\partial\rho/\partial C)_{T,p}$	$D \times 10^5 (\text{cm}^2/\text{s})$
In air			
Ammonia	0.78	+1.07	
Carbon dioxide	0.94	-0.34	
Hydrogen	0.22	+13.4	
Oxygen	0.75	-0.094	
Water	0.60	+0.61	
Benzene	1.76	-0.63	
Ether	1.66	-0.61	
Methanol	0.97	-0.095	
Ethyl alcohol	1.30	-0.37	
Ethyl benzene	2.01	-0.73	
In water			
Ammonia	445	-0.5	2.0
Carbon dioxide	453		1.96
Hydrogen	152		5.85
Oxygen	356		2.5
Nitrogen	468		1.9
Chlorine	617		1.44
Sulphur dioxide	523		1.7
Calcium chloride	750	+0.8	1.188
Sodium chloride	580	+0.7	1.545
Methanol	556	-0.17	1.6
Sucrose	1700		0.5226

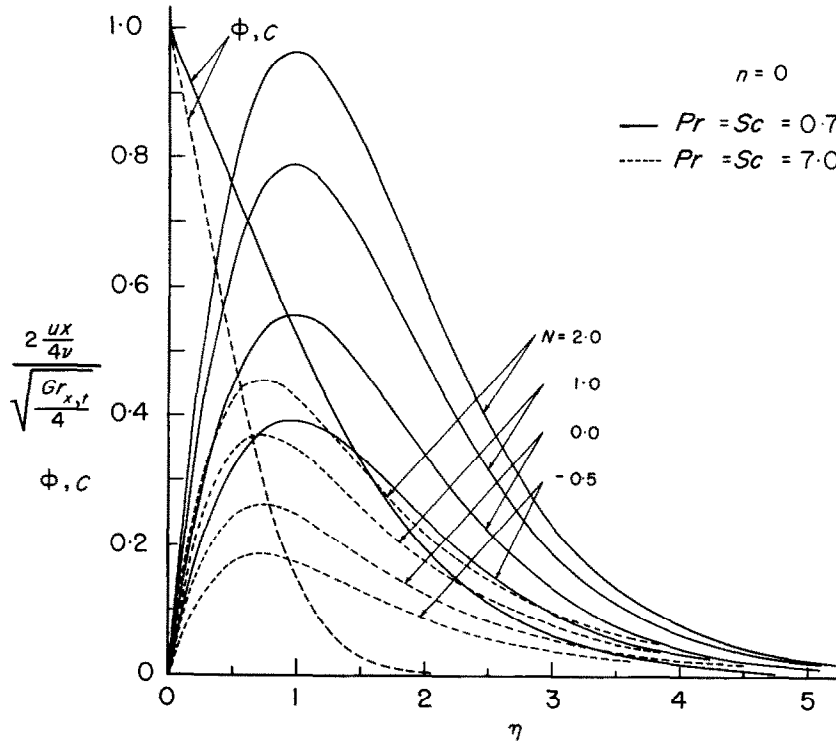


FIG. 1. Flow adjacent to a vertical surface, variation of temperature, concentration and velocity over the boundary region for the Prandtl numbers of air and water and for various conditions ($N = -0.5, 0, 1$ and 2) of the combined buoyancy effects.

nature of the interaction of the multiple contributions to buoyancy. These may aid or oppose and be of different magnitude (N). It is noted that for opposed effects the value of N has a limiting negative value, to assure flow in the positive x direction. For $Pr = Sc$ this value is $N = -1$. The relative physical extent (η) of the two effects in the convection region is governed by the magnitudes of the Prandtl and Schmidt numbers and by their relative values. Another interesting aspect of such flows is that their limits of laminar stability and the nature of disturbance growth may be sharply dependent upon the relative magnitude of the multiple buoyancy effects.

The purpose of the calculations given here is to assess the effects of the parameters Pr, Sc

and N upon the nature of the flow and transport. Since the two most important fluids are atmospheric air and water, the results are limited to those for Prandtl numbers of 0.7 and 7.0. In air the diffusing chemical species of most common interest have Schmidt numbers in the range from 0.1 to 10. Therefore, this range is considered. For water the species H_2, O_2, CO_2 , salt and methanol are perhaps typical of the materials of most interest. A Schmidt number range of 100–500 includes these processes and is considered here, although the value of 1.0 is also included. Table 1 lists nominal values of molecular diffusion parameters for a number of important species.

Finally, for both fluids and ranges of Schmidt number, the flows considered are those adjacent

Table 2. Flow and transport quantities for flows adjacent to isothermal, vertical surfaces. Note that J , L , I_t and I_c are for both sides combined, as for plume flows

Pr	Sc	N	$f''(0)$	$-\phi'(0)$	$-C'(0)$	J	L	I_t	I_c		
0.7	0.1	0.5	0.74221	0.54913	0.19561	2.278	0.469	0.522	1.283		
		1	0.77213	0.56620	0.20656	2.514	0.568	0.538	1.362		
		2	0.80148	0.58125	0.21554	2.698	0.660	0.552	1.425		
	0.5	-0.5	0.64154	0.45891	0.37536	0.661	0.167	0.536	0.495		
		0.5	0.69018	0.50768	0.42747	1.271	0.255	0.483	0.566		
		1	0.69584	0.51152	0.43110	1.300	0.265	0.487	0.572		
		2	0.70185	0.51559	0.43440	1.346	0.276	0.491	0.579		
	0.7	All	0.67890	0.49950	0.49950	1.212	0.235	0.475	0.475		
		0.94	-0.5	0.70824	0.51709	0.59582	1.310	0.275	0.491	0.421	
	5.0	0.5	0.5	0.66880	0.49292	0.57019	1.164	0.221	0.469	0.404	
			1	0.66378	0.48947	0.56659	1.142	0.214	0.465	0.402	
		2	0.65873	0.48588	0.56286	1.119	0.206	0.462	0.399		
			-0.5	0.85198	0.56603	1.29997	1.416	0.366	0.366	0.173	
		10.0	0.5	0.5	0.61725	0.46880	1.11689	1.096	0.185	0.445	0.149
				1	0.58574	0.45026	1.08559	1.044	0.159	0.429	0.145
			2	0.55277	0.42829	1.04962	0.941	0.129	0.406	0.140	
				-0.5	0.90099	0.57552	1.71754	1.425	0.378	0.548	0.114
		7.0	7.0	All	0.45069	1.05411	1.05411	0.539	0.047	0.100	0.100
				1.0	0.5	0.51983	1.16002	0.48743	0.827	0.101	0.110
	1				0.55161	1.19963	0.51188	0.896	0.124	0.114	0.341
100	2		0.58246	1.12348	0.53263	0.953	0.145	0.117	0.355		
			-0.5	0.61831	1.21566	3.33176	0.627	0.075	0.116	0.022	
	0.5		0.38933	0.97593	2.75894	0.453	0.034	0.093	0.018		
			1	0.35748	0.92855	2.65410	0.417	0.028	0.088	0.018	
500	2		0.03244	0.87121	2.53325	0.372	0.022	0.083	0.017		
			-0.3	0.54994	1.14494	5.58873	0.584	0.060	0.109	0.0074	
	0.5		0.36783	0.96154	4.79946	0.473	0.034	0.092	0.0064		
		1	0.32402	0.90298	4.56087	0.438	0.028	0.086	0.0061		
	2	0.27788	0.82889	4.27465	0.383	0.020	0.079	0.0057			

to a surface, at uniform temperature and/or concentration ($n = 0$) and those in a freely rising plume ($n = -\frac{2}{5}$).

RESULTS OF CALCULATIONS FOR $Pr = Sc$

It is convenient to discuss first the nature of the flow and diffusion processes arising from the combined buoyancy effects of thermal ϕ and mass diffusion C in the circumstance of equal Prandtl and Schmidt numbers. It was shown above that, in this event, only f and ϕ (or f and C) need be considered, the combined effect being solely reflected in the definition of η and f and by the value of N . Thus, the well-known thermal natural convection results are adequate. For

vertical surfaces one is particularly interested in the flow and heat transfer parameters $f''(0)$ and $C'(0)$. These are shown in Table 2 for certain values of N over the range of Prandtl numbers of 0.7 and 7.0 for the uniform surface temperature condition.

The effects of the diffusion parameter Pr and the relative magnitudes of thermal and species diffusion (N) are seen in Fig. 1 where the tangential velocity component μ and ϕ ($= C$) are plotted vs. η for $Pr = 0.7$ and 7.0 for $N = -0.5, 0, 1$ and 2.0. The effect of increasing Prandtl number is seen to be the reduction of the velocity level and the thinning of the thermal and species diffusion layer. The value of N has

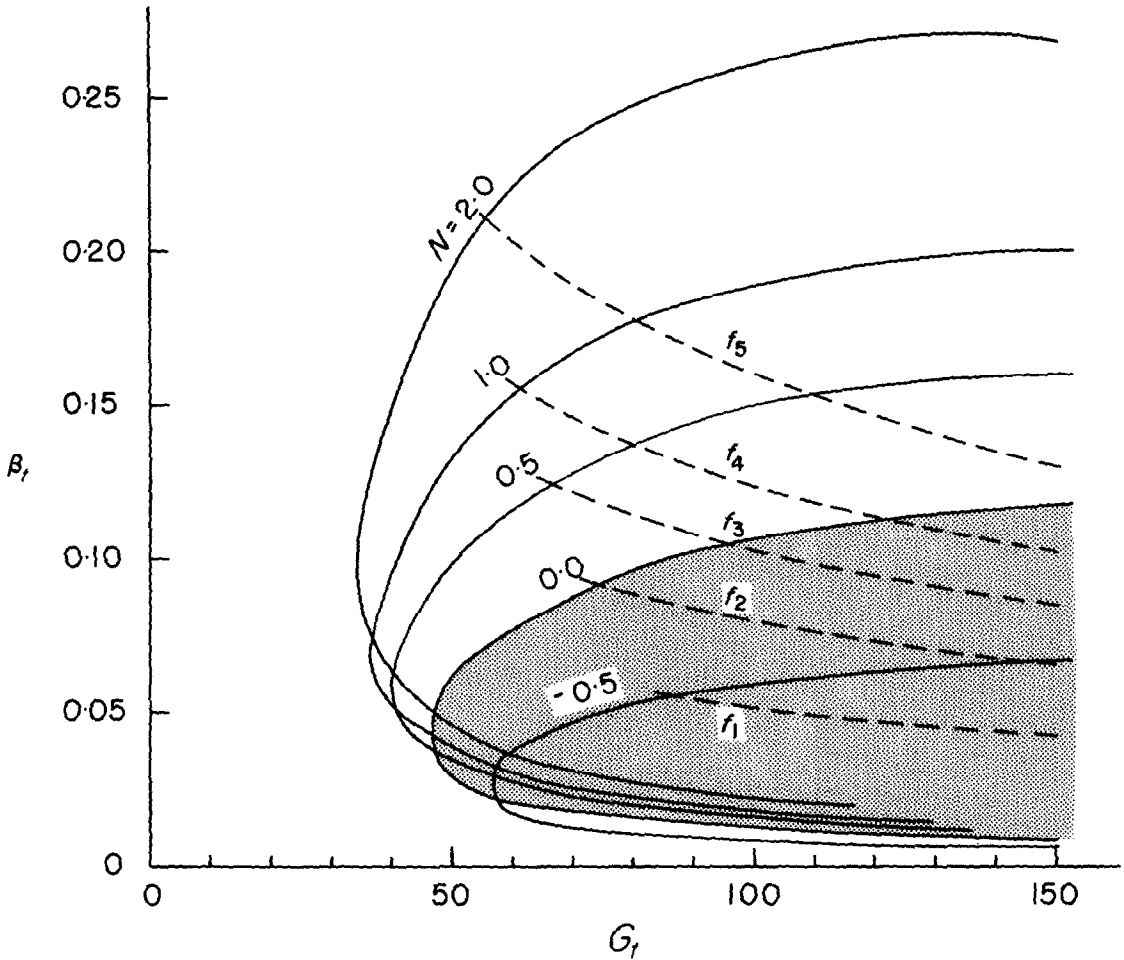


FIG. 2. Neutral stability curves for the flow of a fluid having a Prandtl number of 6.7, with combined buoyancy effects, flowing adjacent to a uniform surface flux vertical surface, $\beta_1 \propto \beta x^2/\nu\sqrt{(Gr_{x,i})}$ and $G_r \propto \sqrt{(Gr_{x,i})}$.

no effect on the ϕ and C distributions in terms of the combined driving force $Gr_{x,i} + Gr_{x,c}$ but changes the magnitude of the velocity, in terms of $Gr_{x,i}$, over the boundary region of unchanged thickness in η but of decreasing physical thickness with increasing N . Transport rates are simply calculated from equations (19) and (20) for $P = Q = 1$ knowing only $-\phi'(0)$ and N .

One of the other principal interests in studying such flows is to determine the effect that the combined agencies of buoyancy have on the

stability and disturbance growth rates for such laminar flows. The result may be shown to be particularly simple for $Pr = Sc$. The formulation of the linear stability analysis proceeds just as before, see, for example Plapp [16], Nachtsheim [17] and Knowles and Gebhart [18]. The generalizing driving force, convection velocity, and convection region extent are written in terms of $Gr_{x,i} + Gr_{x,c}$. For $Pr = Sc$ the disturbance coupling effect through buoyancy becomes either the thermal or species diffusion

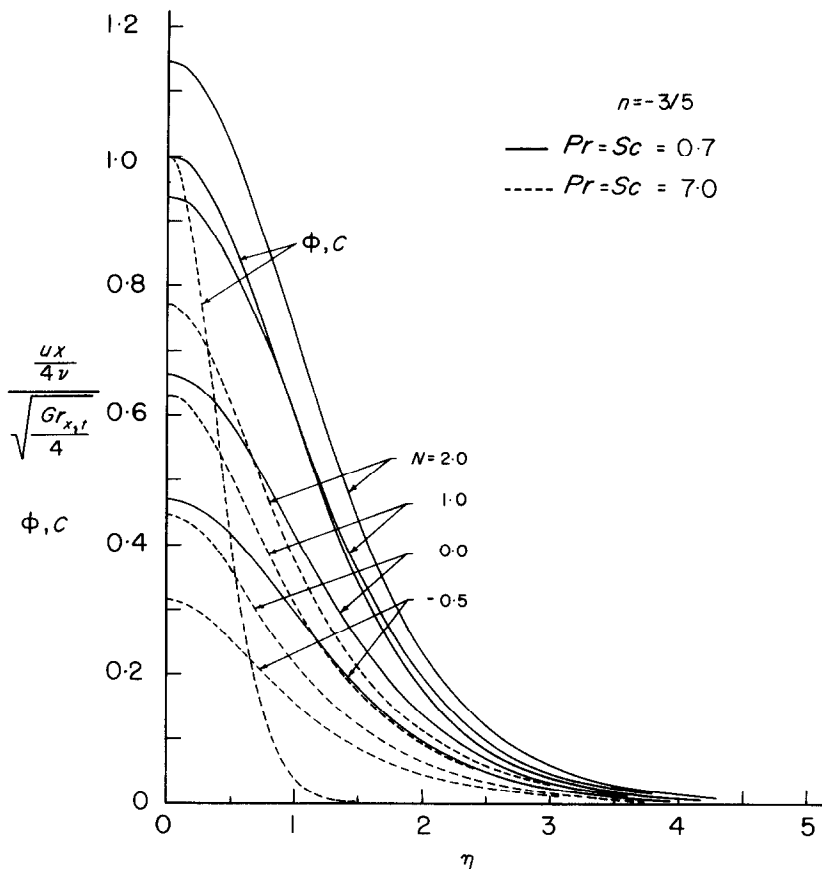


FIG. 3. Plume flows, variation of temperature, concentration and velocity for the Prandtl numbers of air and water and for various conditions ($N = -0.5, 0, 1$ and 2) of the combined buoyancy effects.

term. The only difference then in the Orr-Sommerfeld formulation is the meaning of the local Grashof number, which becomes $Gr_{x,t} + Gr_{x,C}$, and in the wave-length α and frequency parameters β which are similarly generalized in terms of $Gr_{x,t} + Gr_{x,C}$. The base flow functions f and ϕ are unchanged. Therefore, the eigenvalues are also unchanged and the only effect is the necessity to interpret stability limits, growth rates, and likely ranges of transition to turbulence in terms of a Grashof number which is a combination of the two buoyancy effects. Thus the stability planes of Knowles and Gebhart [18], Dring and Gebhart [19] and

Hieber and Gebhart [20] and [21] may be used directly. Conditions for eventual transition to turbulence are predicted by Hieber and Gebhart [20].

A stability plane is shown in Fig. 2 for $Pr = 6.7$ and for a uniform flux condition on a vertical surface. Neutral stability curves are shown for increasing values of N in terms of the frequency parameter β_r , and G_r , expressed in terms of only the thermal buoyancy effect. Thus, we think of the curves as all applying for a given level of thermal effect with an added species diffusion effect whose magnitude, relative thereto, is indicated by N . Increasing this effect destabilizes

Table 3. Flow and transport quantities for plume flows

<i>Pr</i>	<i>Sc</i>	<i>N</i>	<i>f'(0)</i>	<i>J</i>	<i>L</i>	<i>I_t</i>	<i>I_c</i>	
0.7	0.1	0.5	0.73587	3.251	1.339	1.398	2.463	
		1	0.76508	3.628	1.564	1.458	2.658	
		2	0.79178	3.881	1.769	1.508	2.809	
	0.5	-0.5	0.62035	1.343	0.661	1.097	1.177	
		0.5	0.67334	1.954	0.878	1.234	1.361	
		1	0.67891	1.996	0.902	1.251	1.377	
		2	0.68435	2.035	0.926	1.263	1.392	
		All	0.66183	1.862	0.828	1.212	1.212	
	0.94	-0.5	0.68855	1.996	0.929	1.264	1.149	
		0.5	0.65243	1.810	0.793	1.192	1.089	
		1	0.64763	1.782	0.775	1.182	1.081	
		2	0.64273	1.751	0.757	1.171	1.072	
		-0.5	0.79837	2.171	1.217	1.386	0.628	
		0.5	0.60840	1.713	0.683	1.131	0.535	
		1	0.57931	1.620	0.606	1.082	0.518	
		2	0.54805	1.507	0.525	1.024	0.500	
		-0.5	0.83104	2.187	1.271	1.405	0.460	
		0.5	0.59386	1.701	0.661	1.119	0.383	
	7.0	7.0	1	0.55628	1.600	0.572	1.062	0.369
			2	0.51532	1.474	0.477	0.993	0.353
100		All	0.44419	1.059	0.295	0.375	0.375	
		-0.5	0.57468	1.226	0.450	0.431	0.122	
		0.5	0.39095	0.980	0.237	0.350	0.100	
		1	0.36130	0.932	0.207	0.334	0.096	
		2	0.32884	0.874	0.174	0.316	0.092	
		-0.5	0.60360	1.248	0.476	0.440	0.056	
		0.5	0.37628	0.966	0.226	0.344	0.044	
		1	0.33798	0.936	0.192	0.325	0.042	
2	0.29200	0.802	0.147	0.299	0.039			

the flow at a smaller *x* and increases the frequency of the most unstable disturbance. Also shown are the paths along which the frequency is convected which will have the highest amplitude at the condition in *G_t* for which transition is often observed. This appreciable frequency increase is a very interesting effect, since the sensitive frequencies are typically very low in natural convection flow.

It is perhaps initially surprising that stability limits and disturbance growth rates are affected in only this simple way by the additional buoyancy mode. However, for *Pr = Sc*, the additional effect does not cause any change in the forms of the distributions of velocity or temperature or of concentration. Thus, the only effect is a change in the vigor of the flow and this effect is included in the stability equa-

tion when *Gr_x* becomes *Gr_{x,t} + Gr_{x,c}*. The above conclusions show that pure species diffusion driven flows also have the same stability results when *Gr_{x,t}* is replaced by *Gr_{x,c}* and *Pr* is replaced by *Sc*.

However, the above conclusions concerning stability do not apply to flows in which *Pr ≠ Sc*. In this circumstance the thermal and species concentration regions do not completely overlap. The form of the velocity distribution is, therefore, affected by the value of *N*. These combined influences may greatly alter the stability characteristics of such coupled flows.

For convection plume flows resulting from combined buoyancy effects, the results are similar to those discussed above for flow adjacent to surfaces. The *φ* and *C* distributions are equal. Velocities generalized with *Gr_{x,t}*

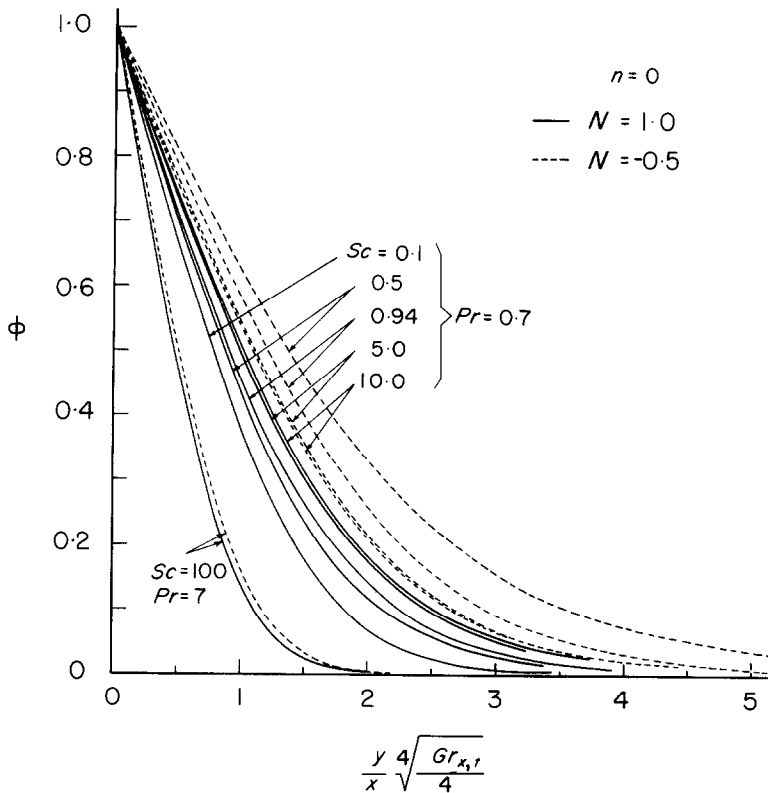


FIG. 4. Flow adjacent to a vertical surface, the effect on temperature distribution of varying Schmidt number at given Prandtl numbers, for $N = -0.5$ and 1 .

depend only on N for any given Prandtl number when generalization is accomplished with $Gr_{x,t} + Gr_{x,c}$. Figure 3 shows the effects of N on flow for $Pr = 0.7$ and 7.0 for $N = -0.5, 0, 1$ and 2 . Various flow and transport characteristics may be determined from equations (22) through (27), using values listed in Table 3 for $Sc = Pr$.

The stability characteristics of laminar flow in such plumes is the same as discussed above for surfaces and the calculations of Pera and Gebhart [22] may be used, interpreted as indicated above for flows adjacent to vertical surfaces. Table 3 lists the various useful parameters of plume flow for equal Schmidt number for Prandtl numbers of 0.7 and 7.0 . These may be used with the appropriate value of N to characterize combined effect flows.

RESULTS OF CALCULATIONS FOR $Pr \neq Sc$

This is the interesting circumstance, in which the thermal and species diffusion regions are inherently of different extents. Thus, both aiding and opposing buoyancy effects alter the form of the distributions and in an amount which is dependent upon their relative magnitude N . For $Sc < Pr$ and $N < 0$ the velocity component u may actually become negative in the outer part of the flow region and for $Pr < Sc$ and $N < 0$ it may be negative in the inner part.

The characteristics of such flows are perhaps best seen by considering numerical solutions of equations (14)–(16) with boundary conditions (17) and (18), for a uniform surface temperature condition ($n = 0$) and for plumes ($n = -\frac{2}{3}$), respectively. Such calculations are very demand-

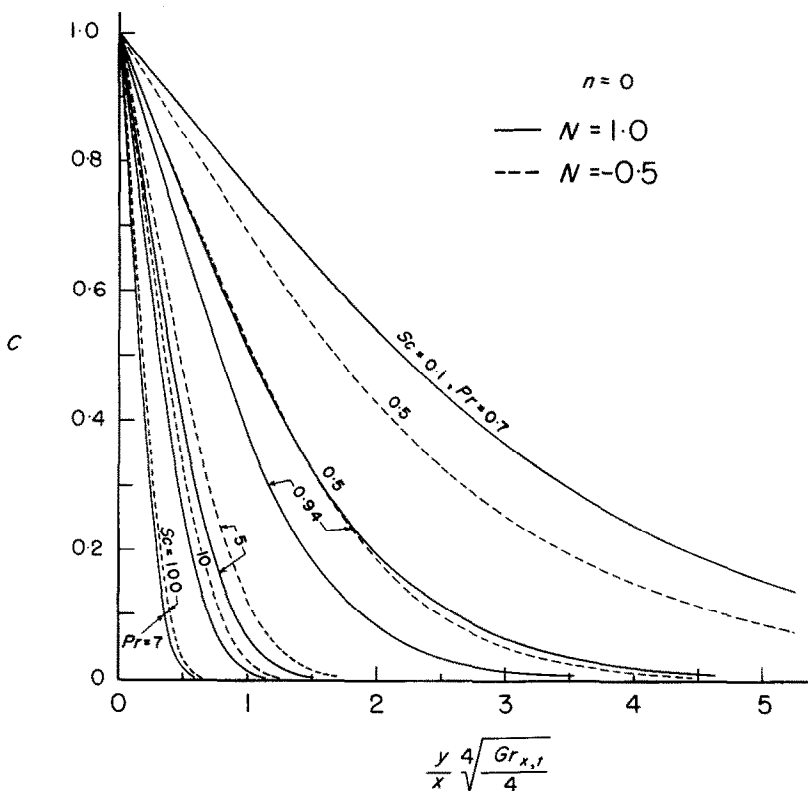


FIG. 5. Flow adjacent to a vertical surface, the effect on species concentration distribution of varying Schmidt number at given Prandtl numbers, for $N = -0.5$ and 1 .

ing and only the nominal Prandtl numbers of air and of water (0.7 and 7.0) are considered, with the important Schmidt number values of 0.1, 0.5, 0.94, 5 and 10 for $Pr = 0.7$ and of 1,100 and 500 for $Pr = 7.0$. For the calculations discussed immediately below P and Q were taken as 1.0. A later discussion considers the advantages of choosing P and Q otherwise.

Solutions were obtained for each of these combinations for $N = -0.5, 0.5, 1$ and 2 , that is, for effects opposing, and various ratios of aiding effects.

The resulting values of various important transport parameters are listed in Tables 2 and 3 for flow adjacent to vertical surfaces and in plumes, respectively. These values may be used

with preceding equations (19)–(27) to calculate heat and chemical species transport, velocity, and plume flow convection quantities.

For flow adjacent to a vertical surface ($n = 0$) the distributions of temperature, concentration and velocity are shown in Figs. 4–6. They are plotted in terms of $\sqrt[4]{(Gr_{x,t})}$ and, therefore, represent the effect in magnitude and in extent of the boundary region, of the species diffusion buoyancy mechanism, for a given level of thermal diffusion caused buoyancy. Results are shown for each Schmidt and Prandtl number for $N = -0.5$ and 1.0 , that is, for the species buoyancy effect of half strength and opposing and for the effects equal and aiding.

Figures 4 and 5, for ϕ and C , indicate the

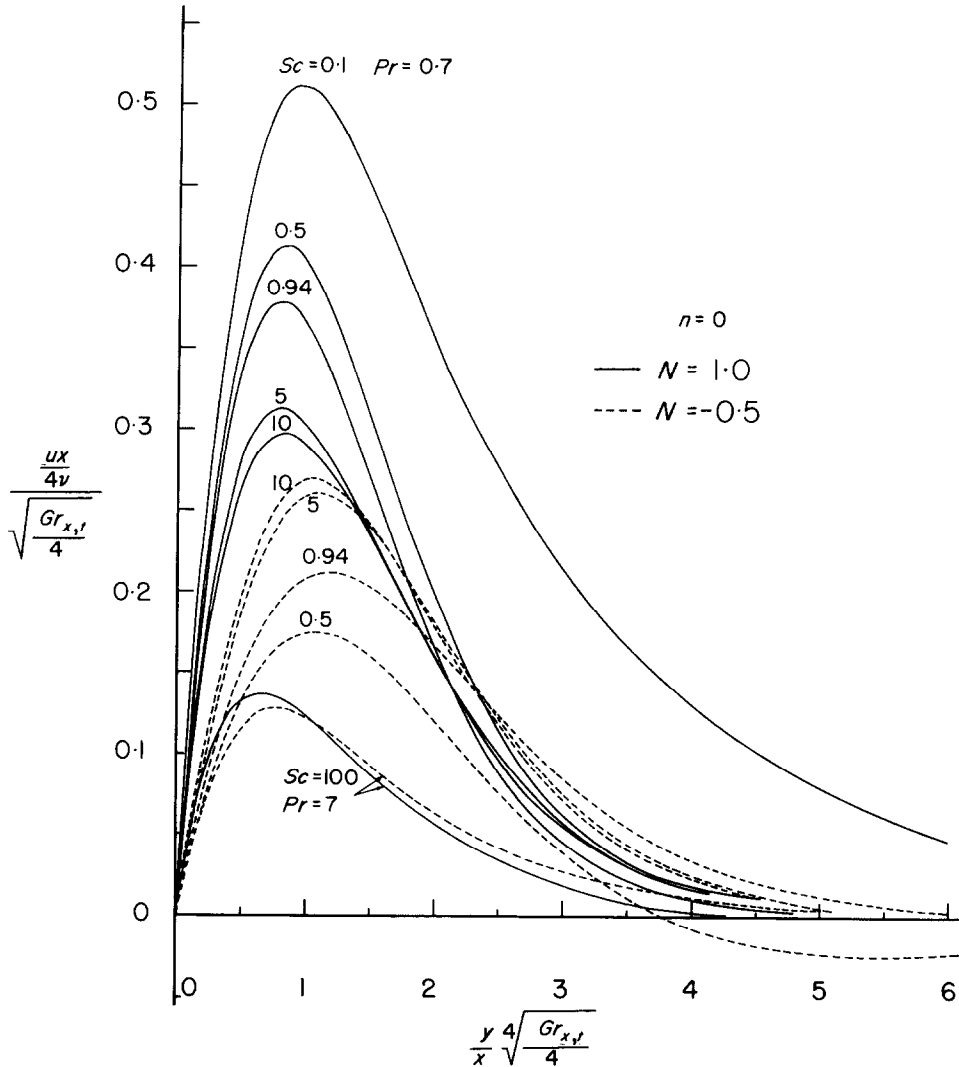


FIG. 6. Flow adjacent to a vertical surface, the effect on flow velocity of varying Schmidt numbers at given Prandtl numbers, for $N = -0.5$ and 1.

thinning effect of large values of both Pr and Sc and the unequal extents of the two regions. Opposed buoyancy effects thicken and aiding ones thin the regions, compared to the flows caused by the purely thermal effect, not shown. The velocity distributions in Fig. 6 show much larger effects, both in magnitude and in extent. The most striking result, for $Pr = 0.7$, indicates

that the Schmidt number effect upon the temperature and velocity levels is reversed for $N = -0.5$ and 1.0. These results follow from a very complicated interaction of the two buoyancy effects, through velocity, to the diffusion mechanisms. For aiding effects, decreasing Schmidt number increases the velocity level and its extent, thus thinning the temperature

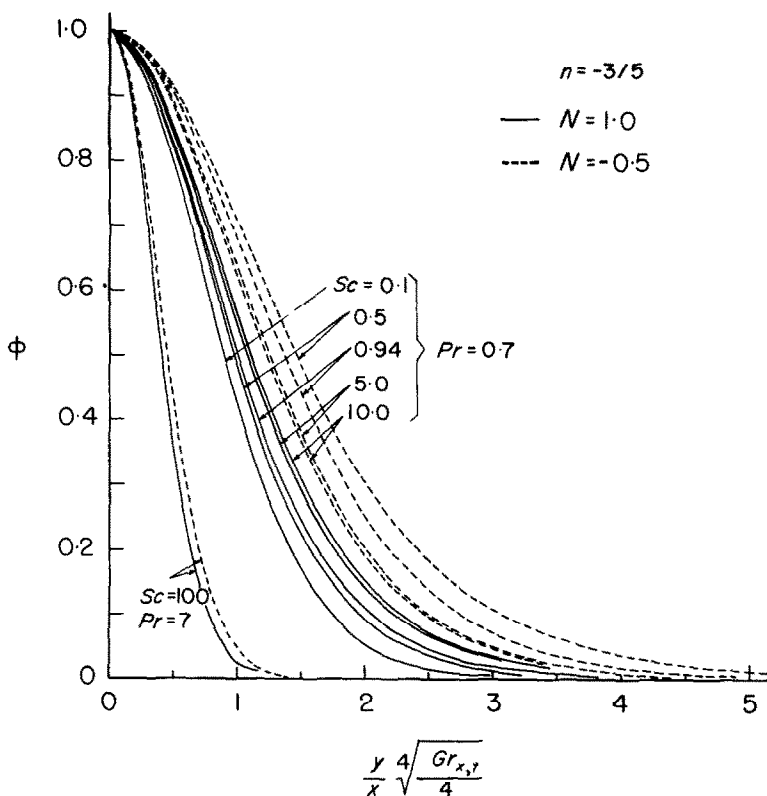


FIG. 7. Plume flows, the effect on temperature distribution of varying Schmidt number at given Prandtl numbers, for $N = -0.5$ and 1 .

region. For opposed effects, velocities are much lower and less for decreasing Schmidt number. Thus, the temperature region becomes thicker.

Another interesting result is the large distortion of the velocity field caused by decreasing values of N . Negative values of velocity are predicted in the outer boundary region for smaller values of Sc for N negative. This results from the incoming flow first experiencing the negative buoyancy effect of the thicker species diffusion layer. The flow is eventually drawn upward by the combined action of thermally caused buoyancy and the shear that results therefrom. These large effects upon the velocity distribution would be expected to have very large effects on the stability of such laminar

flows. In particular the very major distortions shown to result for $Pr \neq Sc$ and opposing effects should drastically reduce stability limits for such flows. Note, parenthetically, that the roles of Pr and Sc may simply be interchanged in interpreting the results. Considerations concerning stability are discussed in more detail later.

Figures 7 and 8 for plume flows ($n = -\frac{3}{5}$) shown similar characteristics: aiding effects thin the thermal region and opposed effects greatly distort the velocity distribution. The reversal of the Schmidt number effect upon the temperature and velocity fields between aiding and opposing buoyancy effects is again found for the same reasons. The very large changes of the velocity

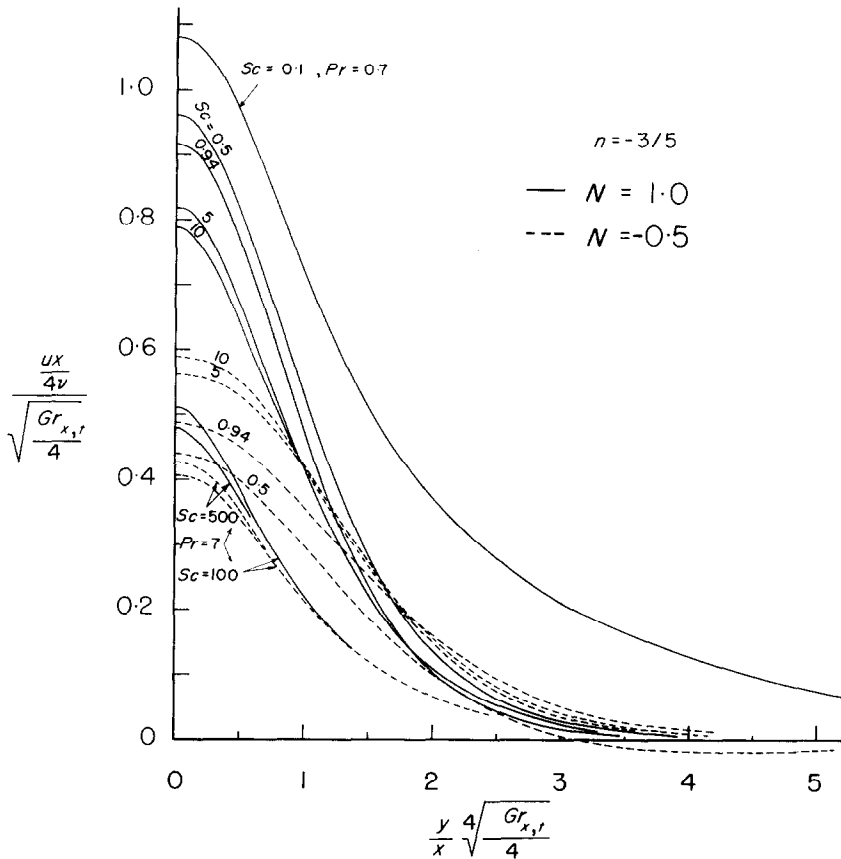


FIG. 8. Plume flows, the effect on flow velocity of varying Schmidt number at given Prandtl numbers, for $N = -0.5$ and 1.

field resulting from both changes in the Schmidt number and in N should again have very large effects on laminar stability limits.

The effects of Schmidt number on the heat and species transfer parameters of equations (19) and (20) are shown as the solid curves in Figs. 9 and 10 for $Pr = 0.7$ and in Figs. 11 and 12 for $Pr = 7.0$, in terms of the parameter N . The values shown at $N = 0$ are the result for thermal buoyancy alone. The results for $N \neq 0$ are very interesting. First, around $N = 0$ heat transfer is not strongly affected by species diffusion although the mass transfer parameter becomes very large. This is because the flow is

induced almost entirely by thermal buoyancy and produces a very effective species diffusion mechanism at very low concentration. For opposed effects ($N < 0$) both diffusion parameters fall off rapidly. The values shown on broken lines for $Sc > Pr$ and N negative were obtained by choosing P and Q in equation (11) different from 1.0. This is discussed subsequently. For aiding effects ($N > 0$) heat transfer is continuously improved by the added impetus to the flow whereas the species diffusion parameters fall toward the values which would pertain for flows driven primarily by species diffusion caused buoyancy. Thus, as $N \rightarrow \infty$

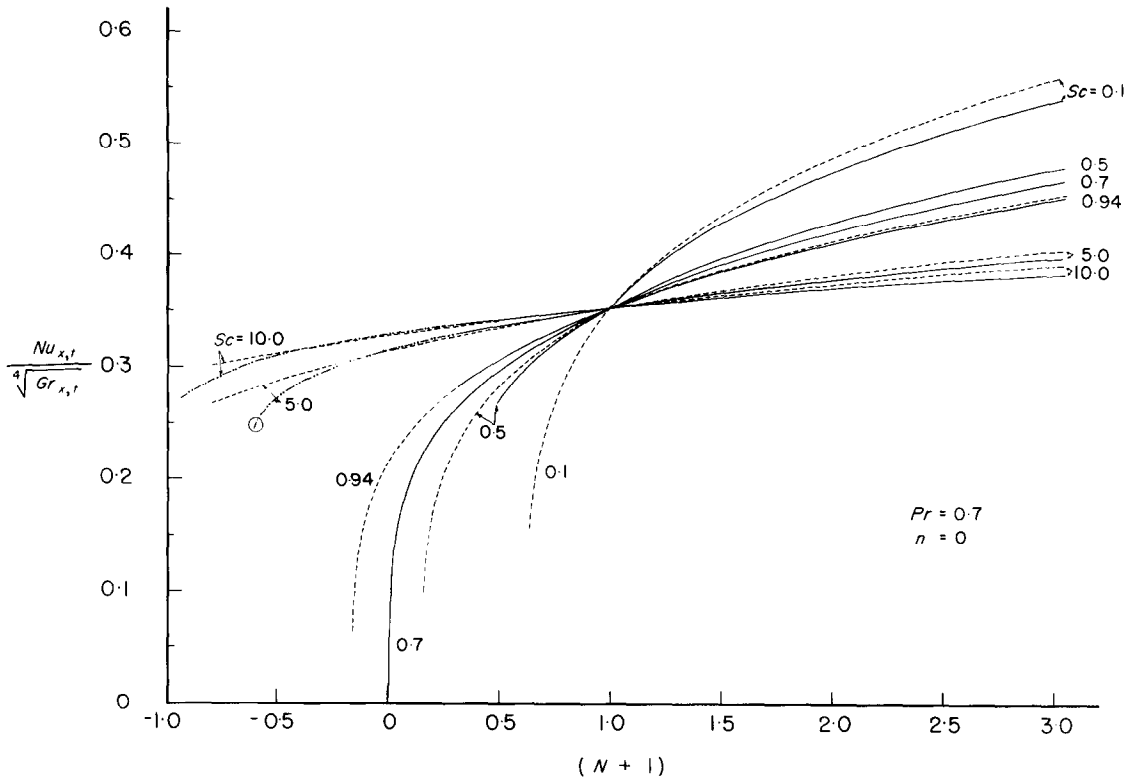


Fig. 9. For flow adjacent to a vertical surface, values of the heat transfer parameter at various Schmidt numbers at a Prandtl number of 0.7, as a function of the relative value (N) of the two buoyancy mechanisms. — Present theory with $P = Q = 1$. - - - Present theory with $P = \sqrt{Sc}$, $Q = \sqrt{Pr}$. — — — Integral method.

the heat transfer parameters become infinite (as for species diffusion at $N = 0$) while species diffusion parameters go to the asymptotic values which pertain for each particular Schmidt number. These characteristics are found for each Prandtl number.

Maximum velocity levels for plumes are similarly shown in Fig. 13. The effect of increasing N is simply to increase the velocity levels. Increasing Schmidt number, at a given value of Pr , merely decreases velocity levels for aiding buoyancy effects since the species diffusion buoyancy effect is confined more deeply in the boundary region. For opposing buoyancy effects the Schmidt number effect on velocity level is reversed, as noted before.

ADDITIONAL OBSERVATIONS CONCERNING FLOWS FOR WHICH $Pr \neq Sc$

The foregoing results, calculated from the appropriate complete equations for such combined buoyancy flows suggest the conditions under which the integral and other approximate methods of analysis may become unrealistic in predictions concerning the nature of the flow or the values of the transport parameters. A comparison of results shows the ways in which these earlier analyses are inappropriate for N much different from zero, and particularly for N increasing negative, and for the Schmidt number appreciably different from the Prandtl number.

But, first, we will consider how, for example,

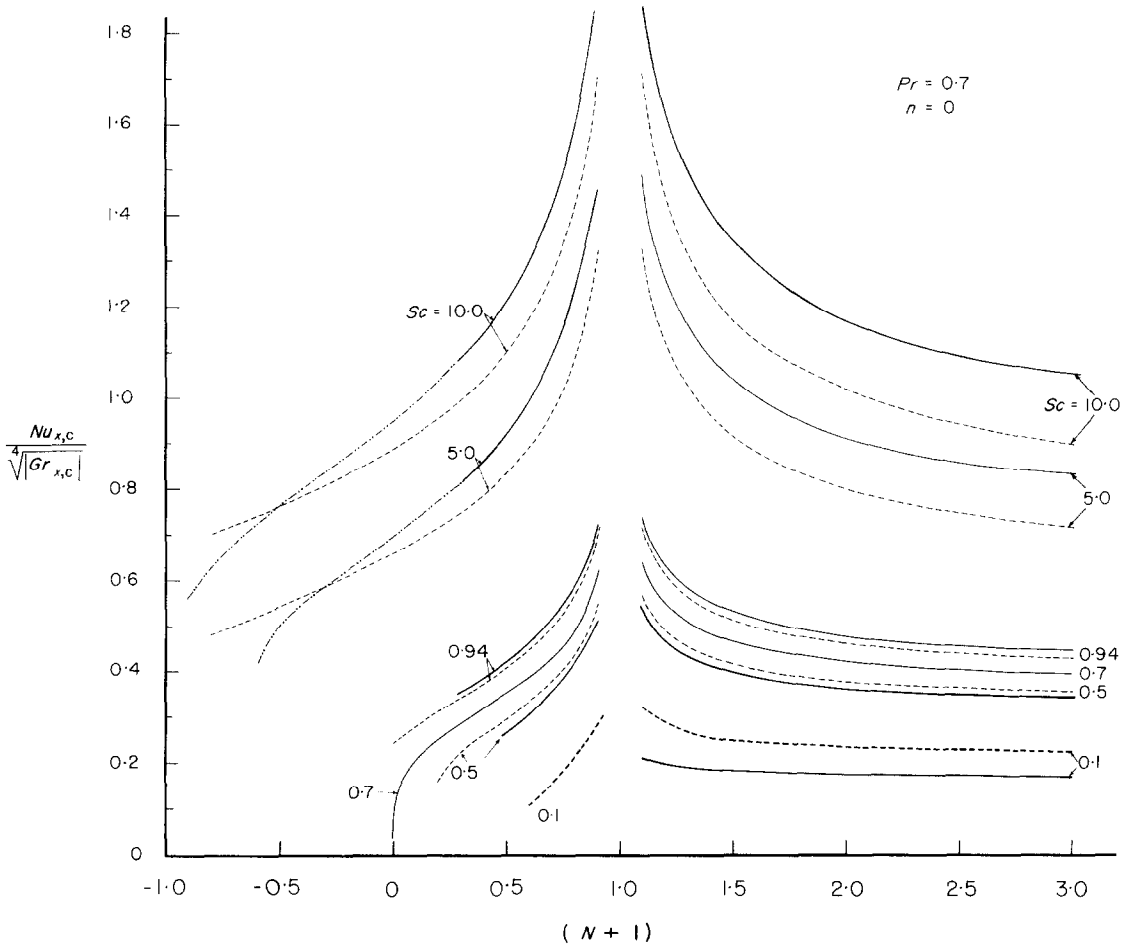


FIG. 10. For flow adjacent to a vertical surface, values of the species transfer parameter at various Schmidt numbers at a Prandtl number of 0.7, as a function of the relative value (N) of the two buoyancy mechanisms. — Present theory with $P = Q = 1$, - - - Present theory with $P = \sqrt{Sc}$, $Q = \sqrt{Pr}$, - · - Integral method.

integral method assumptions must become inappropriate under some conditions. Any *a priori* assumptions of forms of temperature, concentration, and velocity fields should include not only inversions of boundary region velocity levels as a function of Schmidt number as N changes sign, but also of boundary region thicknesses. Thus, the relative thickness of, for example, the velocity and temperature region is not simply a function of Pr , or of Sc . Similarly, the combined buoyancy effect for $N < 0$ greatly

alters the velocity distribution and even produces locally negative values for $Sc \neq Pr$. However, it is shown above that such combined buoyancy effect processes ($Le = 1$) in terms of the generalization employed herein, are the same, for any N , as those arising from a single buoyancy mechanisms, when a combined Grashof number is employed. Thus, in its most appropriate application, such analysis is unnecessary.

The limitations of approximate analysis may

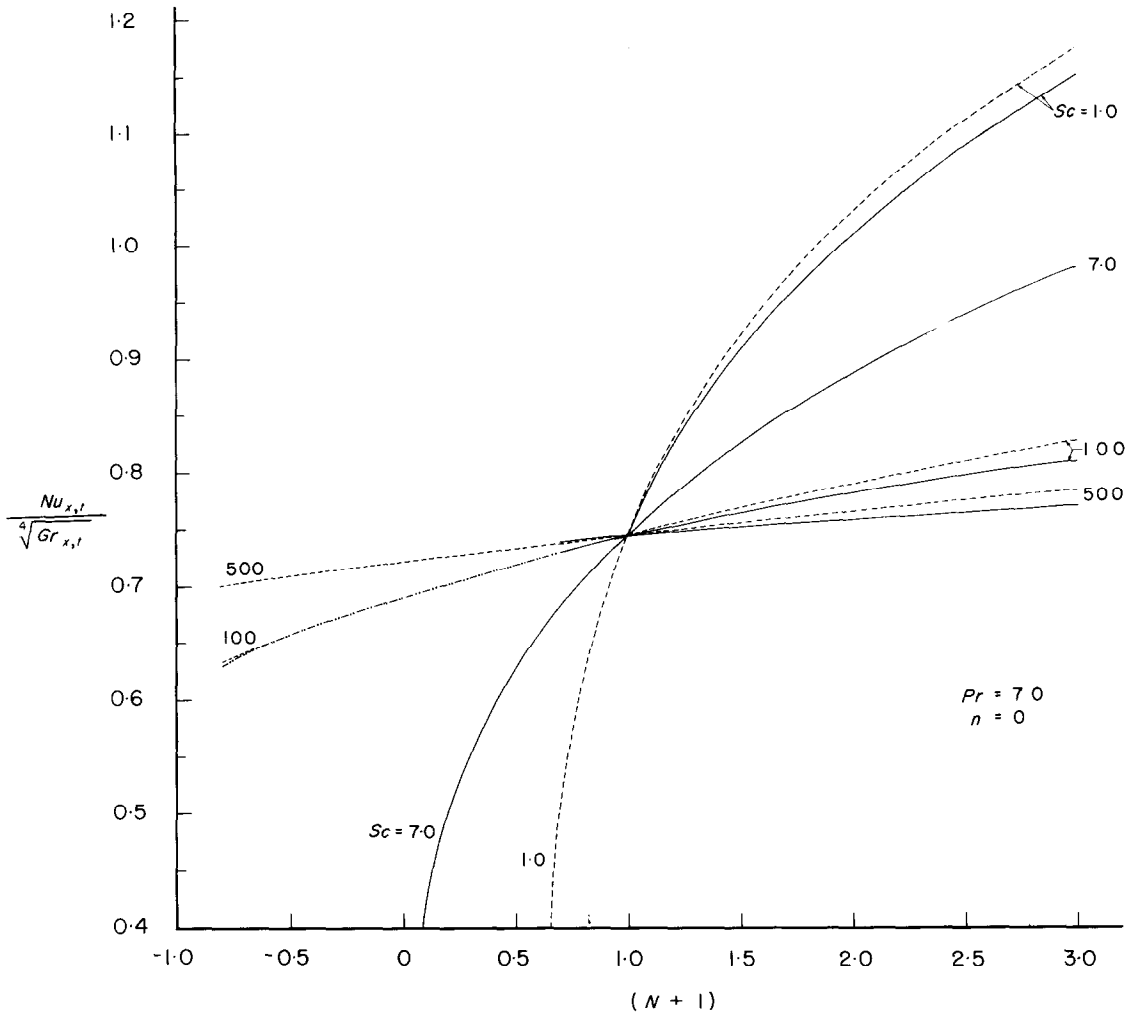


FIG. 11. For flow adjacent to a vertical surface, values of the heat transfer parameter at various Schmidt numbers at a Prandtl number of 7.0, as a function of the relative value (N) of the two buoyancy mechanisms. — Present theory with $P = Q = 1$. - - - Present theory with $P = \sqrt{Sc}$, $Q = \sqrt{Pr}$. ····· Integral method.

be best seen by comparing the resulting predictions of heat and species transport rates with those calculated here. The transport parameters in the present analysis are written as:

$$\frac{Nu_{x,t}}{\sqrt[4]{(Gr_{x,t})}} = \frac{\phi'(0)}{\sqrt{2}} \sqrt[4]{(P + QN)} \quad (19)$$

$$\frac{Nu_{x,c}}{\sqrt[4]{(|Gr_{x,c}|)}} = -\frac{C'(0)}{\sqrt{2}} \sqrt[4]{\left(\frac{P + QN}{N}\right)} \quad (20)$$

where values of $-\phi'(0)$ and $-C'(0)$ are tabu-

lated and plotted as functions of Pr , Sc and N .

The results of most past analyses and correlation attempts may be written collectively in the following forms:

$$\frac{Nu_{x,t}}{\sqrt[4]{(Gr_{x,t})}} = C \sqrt[4]{Pr} \sqrt[4]{\left[1 + \left(\frac{Pr}{Sc}\right)^{\frac{1}{2}} N\right]} \quad (28)$$

$$\frac{Nu_{x,c}}{\sqrt[4]{(|Gr_{x,c}|)}} = C \sqrt[4]{Sc} \sqrt[4]{\left(\left|\frac{(Sc)}{Pr}\right|^{\frac{1}{2}} + N\right)} \quad (29)$$

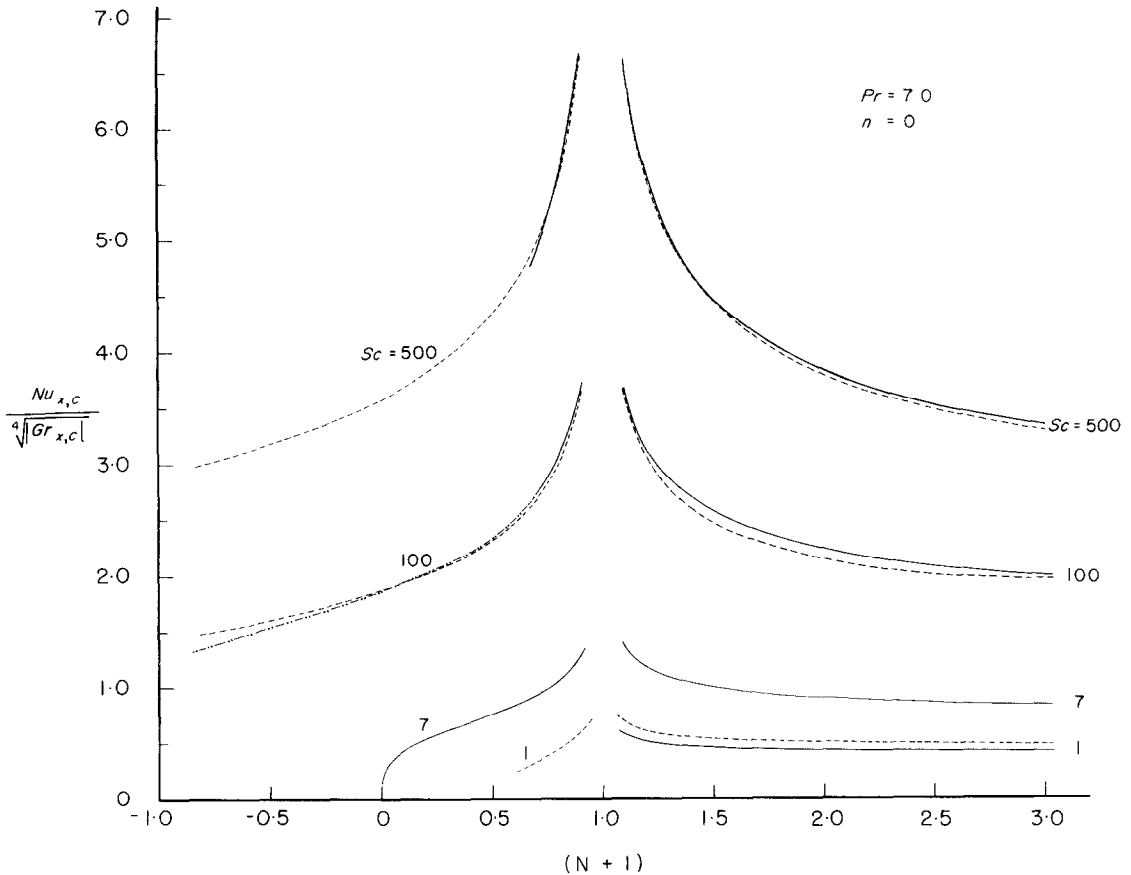


FIG. 12. For flow adjacent to a vertical surface, values of the species transfer parameter at various Schmidt numbers at a Prandtl number of 7.0, as a function of the relative value (N) of the two buoyancy mechanisms. — Present theory with $P = Q = 1$. - - - Present theory with $P = \sqrt{Sc}$. ··· Integral method.

where the factors $\sqrt[4]{Pr}$ and $\sqrt[4]{Sc}$ are the conventional approximation employed in natural convection adjacent to vertical surfaces, and C is a constant independent of Pr , Sc and N . In order for these relations to agree with the results of exact boundary layer analysis for $Pr = Sc$ and $N \rightarrow 0$ and as $N \rightarrow \infty$, the value of C must be 0.387 for $Pr = 0.7$ and 0.458 for $Pr = 7.0$. The comparison is made in terms of these values of C , to remove the conventional approximation in terms of $\sqrt[4]{Pr}$. Thus the comparison considers only the form of the N dependence on the right hand sides of these relations.

This comparison is made by plotting the right

hand sides of equations (28) and (29) as the dashed curves on Figs. 9–12 for $Pr = 0.7$ and $Sc = 0.1, 0.5, 0.94, 5$ and 10 and for $Pr = 7.0$ and $Sc = 1, 100$ and 500 . Recall that the solid and broken curves are the similar quantities of equations (19) and (20), calculated from the exact analysis.

Note from equations (28) and (29) that these approximate expressions have a lower limit of $N > -(Sc/Pr)^{\frac{1}{2}}$. The boundary layer formulation given here does not have this kind of limit. The values of P and Q in equations (19) and (20), and in equation (11), may be chosen as any pair of values which are consistent with physical realism.

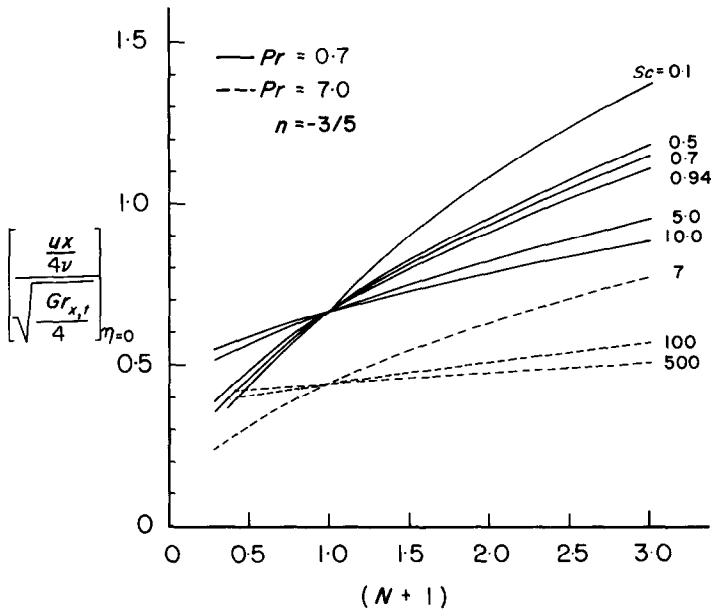


FIG. 13. For plume flow, values of the midplane flow velocity parameter at various Schmidt numbers at Prandtl numbers of 0.7 and 7.0 as a function of the relative value (N) of the two buoyancy mechanisms.

Figures 9–12 show the inaccuracies of the approximate results. The disagreement always increases as the Schmidt number diverges from the Prandtl number. Errors become large for $Sc < Pr$. This is particularly true for species diffusion, see Figs. 10 and 12, except for $Sc \gg Pr > 1$ when the velocity layer is much thicker than both diffusion layers. For $Sc < Pr$, errors in the species transfer parameter may be 35 per cent.

Figures 10 and 12 indicate that the approximate result also becomes unreliable as N becomes sufficiently large negative that the opposing buoyancy effect is appreciable. The trends of the two results become quite different. The physical effect which arises, for example for $Sc > Pr$, is very interesting. Figure 14 shows velocity distributions for $Sc = 5.0$ and $Pr = 0.7$ for $N = 2.0, 0, -1.0$ and -1.6 . For $Sc > Pr$ the opposing species diffusion layer lies near the surface. As N becomes increasingly negative, this negative buoyancy effect alters even the basic form of the velocity distribution, $N =$

-1.6 is the location of the circle in Fig. 9. However, the thicker thermal region induces an outer upflow which still persists, even when the negative buoyancy effect is very large. Clearly the kinds of approximate analyses done to date do not anticipate such complexities and their predictions will begin to fail much before the reasonableness of boundary layer analysis becomes suspect.

CONCLUSIONS

The foregoing development casts multiple buoyancy mechanism laminar natural convection flows into a convenient similarity form. This formulation applies to all two-dimensional and vertical boundary layer flows which admit similarity in single buoyancy mechanism flows.

Many different and complicated boundary conditions fall into this class and the combined problem admits of many additional possibilities.

The calculations indicate many of the complicated interactions which arise in such flows, including local flow reversals and complicated

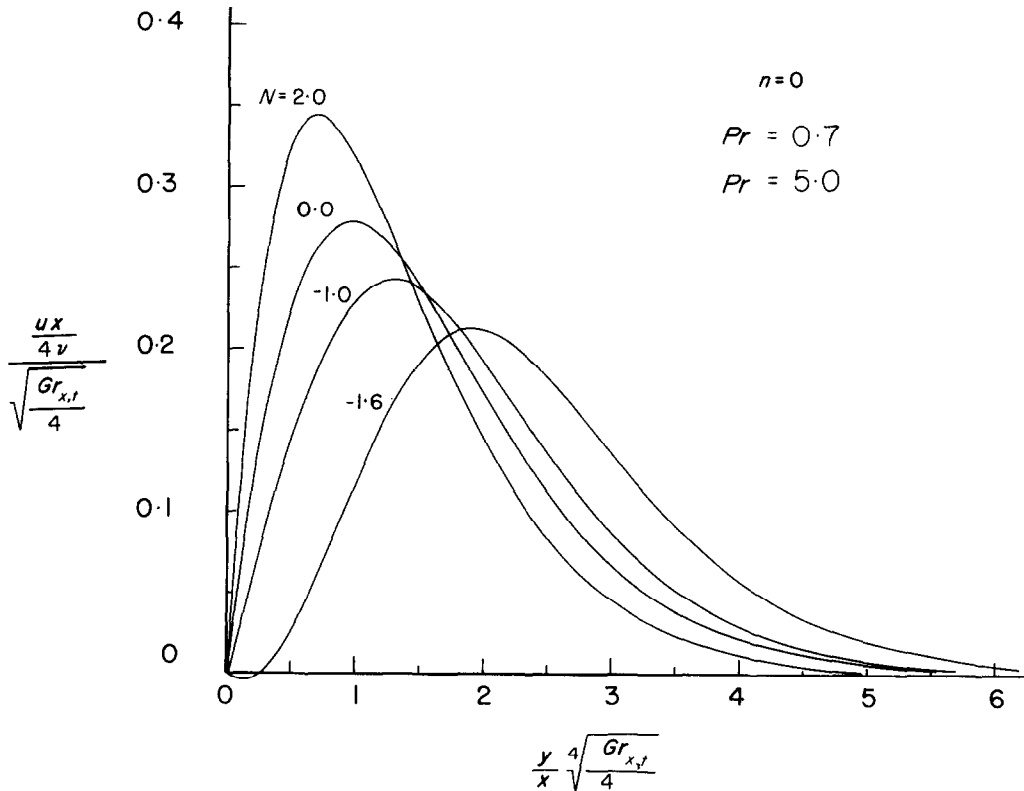


FIG. 14. Velocity distribution adjacent to a vertical surface for a Prandtl number of 0.7 and Schmidt number of 5.0, for species diffusion buoyancy aiding and opposed to that arising from thermal diffusion.

relations of boundary regions thicknesses. The comparisons of transport parameters with those arising from earlier approximate considerations indicate the inaccuracies and limits of the latter. The results of the approximate treatments do not represent the details of the flow and diffusion and fail in the prediction of the transport parameters for conditions away from the most single additive.

No calculations were carried out to assess the effects of combined buoyancy mechanisms on the stability characteristics of laminar flows. For the Prandtl and Schmidt numbers different there certainly will be large effects and these will likely be very different for aiding and opposing effects. This may be anticipated from the forms of the velocity distributions in Figs.

6, 8 and 14. Opposed effects greatly distort the distributions.

However, such calculations, from the relevant Orr-Sommerfeld equations, will be very difficult and demanding. It is clear from past calculations and experiments, [14] and [22], that the coupling between temperature and velocity disturbances, through buoyancy, is a first order effect in such natural convection flows. With combined thermal and species diffusion buoyancy there is an additional coupled mechanism. This adds considerable complexity to the calculation circumstances.

The effect of multiple buoyancy mechanisms upon laminar stability is seen to be very simple for all flows for which the Prandtl and Schmidt numbers are equal. However, multiple buoyancy

mechanisms in general result in flows which are likely to have quite complicated and variable stability characteristics. Analysis of such problems must include the coupling of disturbances through buoyancy.

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NATURE DES MOUVEMENTS VERTICAUX DE CONVECTION NATURELLE RÉSULTANT DE LA COMBINAISON DES EFFETS D'ARCHIMÈDE PAR DIFFUSION THERMIQUE ET MASSIQUE

Résumé—Cet article concerne les écoulements laminaires dans les fluides dus à l'interaction des forces de gravité et des différences de densité causées simultanément par la diffusion d'énergie thermique et des espèces chimiques. Le niveau de concentration des espèces est supposé faible comme dans de nombreux processus dans l'eau et l'air atmosphérique. Les approximations usuelles de Boussinesq conduisent à un système d'équations qui ont des solutions à forme de similitude pour des effets combinés d'Archimède dans le cas d'écoulements verticaux adjacents à des surfaces ou en panache. Cette similitude a la même forme que pour les écoulements à simple effet d'Archimède. Les équations réduites sont intégrées pour l'air et l'eau avec différentes valeurs pratiques du nombre de Schmidt et pour des effets d'Archimède de même sens ou en opposition. Les résultats montrent des effets intéressants sur les vitesses, les transferts de masse et de chaleur et sur la stabilité laminaire. Une comparaison des résultats avec ceux de l'analyse par la méthode intégrale montre les limites et les défauts de ces calculs approchés dans des écoulements plus compliqués à effets combinés d'Archimède.

DIE NATUR SENKRECHTER NATÜRLICHER KONVEKTIONSSTRÖMUNGEN. DIE SICH
AUS KOMBINIERTEN AUFTRIEBSEFFekten DER WÄRME- UND STOFFDIFFUSION
ERGEBEN

Zusammenfassung—Diese Arbeit befasst sich mit laminaren Strömungen, die in Fluiden entstehen durch die Wechselwirkung von Gravitationskräften und von Dichteunterschieden, die durch gleichzeitigen Wärme- und Stoffaustausch hervorgerufen werden. Konzentrationsunterschiede wurden klein angenommen, wie es typisch ist für viele Vorgänge in Wasser und atmosphärischer Luft. Die üblichen Boussinesq-Näherungen liefern einen Gleichungssatz, der für gemischte Auftriebseffekte Ähnlichkeitslösungen hat für senkrechte Strömungen längs Oberflächen und in Wassersäulen. Die Ähnlichkeitslösungen sind von derselben Form, wie sie für Strömungen mit einfachen Auftriebseffekten existieren. Die Integration der resultierenden Gleichungen erfolgte für Luft und Wasser für zahlreiche, in der Praxis interessierende Werte der Schmidt-Zahl und für Auftriebseffekte, sowohl in als auch gegen die Strömungsrichtung. Die Ergebnisse zeigen viele interessante Auswirkungen auf Geschwindigkeit, Wärme- und Stoffübergang und laminare Stabilität. Ein Vergleich der Ergebnisse mit denen integraler Methoden zeigt die Grenzen und die Gründe für das Versagen solcher Näherungsberechnungen in komplizierten Strömungen mit gemischten Auftriebseffekten.

ХАРАКТЕР ЕСТЕСТВЕННОЙ КОНВЕКЦИИ В ВЕРТИКАЛЬНЫХ ПОТОКАХ
КАК РЕЗУЛЬТАТ СОВМЕСТНОГО ВЛИЯНИЯ СИЛ ПЛАВУЧЕСТИ,
ВЫЗВАННЫХ ПЕРЕНОСОМ ТЕПЛА И МАССЫ

Аннотация—В данной статье рассматриваются возникающие ламинарные потоки за счёт взаимодействия гравитационной силы и разности давлений, вызванные одновременным переносом тепла и массы. Предполагают, что концентрация диффундирующих компонент мала, что типично для многих процессов в воде и в атмосфере. Обычные приближения Буссинеска, как показано, имеют автомодельные решения для совместных эффектов сил плавучести, для вертикальных течений вблизи поверхностей и в струях. Это решение имеет вид, аналогичный тому, который был найден для случая воздействия одной силы одной природы. Окончательные уравнения проинтегрированы для воздуха и воды при различных значениях диффузионного числа Прандтля и для случаев совпадающих и противоположно направленных подъёмных сил. Результаты показывают интересные влияния сил плавучести на скорость тепло- и массообмена и на устойчивость ламинарного течения. Сравнения результатов с полученными интегральными методами показывает ограничения и причины неудачи этих приближенных расчётов.